On the Forms of Sextic Scrolls of Genus One.

By Virgil Snyder.

1. The following theorems, which were established in my previous paper on unicursal sextic scrolls, will be made use of:

(1.) The complete nodal curve is of order 9.
(2.) Every generator cuts four others.
(3.) Every curve lying on the surface and such that a single generator passes through each point is of genus 1.

The method employed throughout will be that of algebraic correspondence between the points of two curves. If both curves are unicursal, the correspondence must be elliptic; if the curves are elliptic, the correspondence must be rational.

§1.—*General Form and* (3, 3) *Correspondence.*

2. The general sextic scroll of genus 1 can be obtained by joining corresponding points of two binodal quartic curves by means of straight lines. The quartics must have the same characteristic and must intersect in two points which are self-corresponding in the (1, 1) relation connecting the points of the two curves. Varieties exist according as two, one or no plane exists containing three generators.

The equations may be written in the form

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z}{z_1},$$

in which $$x_0 = \wp(u), \; y_0 = \wp'(u), \; x_1 = \frac{A_1 \wp(u) + B_1 \wp'(u)}{C \wp(u) + D \wp'(u)},$$

with similar expressions for y_1, z_1 having the same denominator. This is type I.

The planes of the two directrix curves may intersect in a double generator. The residual nodal curve is now of order 8; it has a triple point lying on the double generator. Type II.

3. The line of intersection may be a triple generator. The nodal curve must now break up into two triple lines since any plane section which passes through the triple generator cuts a non-singular cubic from the surface, hence it cannot meet the nodal curve except on the triple generator. Two cases exist according as the directrices are skew or coincident. The general equation of the first type is

$$z^3 f_3(x, y) + z^2 wx f_2(x, y) + zw^2 x^2 f_1(x, y) + cw^3 x^3 = 0,$$

type III, and of the second is

$$x^3 \phi_3(x, y) + x^2 \phi_2(x, y)(yz - axw) + x\phi_1(x, y)(yz - axw)^2 + b(yz - axw)^3 = 0,$$

type IV. The line $x = 0$, $z = 0$ is the triple generator in each case.

These two surfaces can be most easily generated as follows: Consider a line $x = 0$, $y = 0$ and a non-singular cubic curve $c_3 = 0$ lying in the plane $z = 0$. Let a pencil of lines in $z = 0$, whose vertex is not on c_3, be made projective with the points of the line. Any line of the pencil will cut c_3 in three points. Connect each of these points with the point on the directrix which corresponds to the given line of the pencil. The connecting lines will generate a sextic scroll having the line x, y for triple directrix and contained in a linear congruence. When the vertex of the pencil is at the point in which the directrix cuts the plane of the cubic, the congruence becomes special. These surfaces contain an infinite number of non-singular cubic curves, all of which can be cut from the same cone.

4. A more general correspondence may be established as follows: Let a series of conics be passed through four fixed basis points, three of which lie on the cubic. The equation of the pencil will be $c_3\mu + \varkappa_2 = 0$. Any conic of the pencil will cut the cubic in three points besides the basis points. Connect each of these three points with the point $x = 0$, $y = 0$, $z = \mu$ by means of straight lines. The resulting sextic scroll will have the line x, y for triple directrix.

The residual nodal curve is of order 6; it cuts the triple line 3 times. Each generator cuts the sextic in two points.* Type V.

When two points of intersection of the cubic and a conic are collinear with the origin and the correspondence is such that this conic is associated with the origin, a double generator exists. The nodal curve is now a quintic which Salmon has called of the first kind. It intersects the triple directrix twice and the double generator three times. Type VI.

5. Finally, by making a similar correspondence between the points of a line and those of a binodal quartic curve, a scroll can be defined having two double generators. The residual nodal curve is a quartic of the second kind which cuts the triple directrix in one point. Type VII.

6. Two more types, analogous to III and IV, can be obtained by setting up an elliptic (3, 3) correspondence between the points of two straight lines which may be skew or coincident. If $\frac{y}{x} = \lambda$ and $\mu = \frac{z}{w}$, then $f(\lambda^3, \mu^3) = 0$ must have three finite double elements. If $\lambda = \frac{y}{x}$, $\mu = \frac{xz - ayw}{x}$, then $\sum_{r=0}^{3} \phi_{6-2r}(\lambda)\, \mu^r = 0$ must have three finite double elements. The former is type VIII and the latter is type IX.

§2.—*Scrolls containing a Multiple Conic.*

7. Let a conic and a line which intersects it in one point be put in (3, 2) correspondence in such a way that the point of intersection is a self-corresponding double element.

Let the equations of the line be

$$x = 0, \quad y = 0, \quad z = \mu,$$

and of the conic

$$x = \lambda, \quad y = \lambda^2, \quad z = 0.$$

* This curve is the cuspidal edge of the planar developable $at^4 + bt^3 + \ldots = 0$, in which $a, b, \ldots$ are linear functions of x, y, z. See Salmon's "Geometry," p. 296.

A generator is then defined by

$$\frac{x}{\lambda} = \frac{y}{\lambda^2} = \frac{\mu - z}{\mu},$$

from which

$$\lambda = \frac{y}{x}, \quad \mu = \frac{yz}{yw - x^2}.$$

The most general relation between λ and μ of the form defined is

$$\lambda^2 f_3(\mu) + \lambda\mu f_2(\mu) + \mu^2 f_1(\mu) = 0,$$

from which, after removing the factor y^2,

$$f_3(yz, yw - x^2) + xzf_2(yz, yw - x^2) + x^2z^2 f_1(yz, yw - x^2) = 0,$$

which is the general equation of the surface. Since the point of intersection, regarded as a point on the conic, has one non-coincident corresponding point on the line, the latter is a simple generator and double directrix. The triple line and the triple conic make up the whole nodal curve. Type X.

If the curve and the line be put in (3, 3) correspondence having the point of intersection for a self-corresponding triple point, the line will be a triple directrix but not a generator. The equation is

$$\lambda^3 f_3(\mu) + \lambda^2\mu f_2(\mu) + \lambda\mu^2 f_1(\mu) + a\mu^3 = 0,$$

or, after removing the factor y^3,

$$f_3(yz, yw - x^2) + xzf_2(yz, yw - x^2) + x^2z^2 f_1(yz, yw - x^2) + az^3x^3 = 0.$$

Type XI. These are the only forms having triple conics.

8. The next series is that in which the conic is double and the line multiple. Suppose λ, μ have the same meaning as above and the correspondence be (2, 2). The form of the equation is

$$y^2z^2\phi_2(x, y) + yz(yw - x^2)f_2(x, y) + (yw - x^2)^2\psi_2(x, y) = 0.$$

The line x, y is a double generator and a double directrix. Any plane through it will contain two other generators which intersect on the double conic. The residual nodal line is an additional double generator lying in the tangent plane through the fourfold line and consecutive to the latter. Any plane section of the surface cuts from the nodal line a fourfold point which counts as seven double points. It is a tacnode with two simple branches passing through it in different directions. Type XII.

If the correspondence be (2, 3) with a double element, the form of the equation is

$$\mu^2 f_3(\lambda) + \lambda\mu f_2(\lambda) + \lambda\phi_2(\lambda) = 0,$$

in which f_3, f_2, ϕ_2 are restricted so that a double element exists. In particular, if ϕ_2 is of the form u^2 and u is a factor of f_2, the double generator lies in the plane of the conic.

The triple directrix counts as single generator. The fourfold line, the double conic and the double generator make up the whole nodal curve. The equation is

$$yz^2 f_3(x, y) + yzf_1(x, y)\,\phi_1(x, y)(yw - x^2) + (yw - x^2)^2\,\phi_1^2(x, y) = 0. \quad \text{Type XIII.}$$

Let there be (2, 4) correspondence between the points of the conic and the straight line such that the point of intersection is a self-corresponding double element and having one other double element. The directrix is now fourfold and is not a generator. In particular, if the nodal generator lie in the plane of the double conic, the equation is of the form

$$z^2 f_4(x, y) + z(xw - y^2) f_2(x, y)\,.\,\phi_1(x, y) + (xw - y^2)^2\,\phi_1^2(x, y) = 0. \quad \text{Type XIV.}$$

It will be observed that the last three equations are all of the same type; the scrolls, however, are essentially different. In XII, two generators issue from each point of the directrix, and for two different points one of the generators coincides with the directrix; in XIII, three generators issue from each point and only one generator coincides with the directrix; finally, in XIV, four generators issue from each point, and none of them coincides with the directrix. These are the only types having a double conic and a multiple directrix which intersects it.

9. Consider a (2, 2) correspondence between the points of a conic and of a line which does not intersect it. Let the equations of the conic be

$$x = \lambda, \quad y = \frac{1}{\lambda}, \quad z = 0$$

and of the line $\qquad x = 0, \quad y = 0, \quad z = \mu.$

The equations of the line joining λ to μ give

$$\lambda^2 = \frac{y}{x}, \quad \mu = \frac{\lambda z}{\lambda - x}.$$

Substitute these values in a general (2, 2) correspondence between λ, μ. The result may be written in the form

$$(f_2(x, z, w)x + \phi_2(x, z, w)y + axy^2)^2 = xy(\psi_2(x, y, w) + yf_1(x, z, w))^2,$$

from which the residual curve of order 6 is directly evident. The sextic cuts the double directrix twice and the conic twice. Each generator cuts the sextic twice. Type XV.

When the values of λ which correspond to $\mu = 0$ give points on the conic which are collinear with the origin the line joining these points is a double generator. The residual quintic curve cuts the double directrix but once. The expression for the (2, 2) correspondence is now of the form

$$\phi_2(\lambda)\mu^2 + f_2(\lambda)\mu + c(\lambda^2 - \varkappa^2) = 0,$$

from which the equations of the surface and of the nodal quintic can be at once obtained. Type XVI.

10. Two conics lying in different planes but having two points of intersection, generate a sextic scroll when put in (2, 2) correspondence with each point of intersection as a single self-corresponding point.

Let the equations of the conics be

$$x = \varkappa\mu, \quad y = \mu^2, \quad z = 0,$$
$$x = 0, \quad y = \lambda^2, \quad z = \lambda.$$

The equations of the line joining the point λ to the point μ are

$$\lambda x = \mu\lambda - \mu z, \quad \varkappa y - \varkappa\lambda z = \mu x.$$

The parameters λ, μ are connected by the relation

$$a\mu^2\lambda + b\mu\lambda^2 + c\lambda^2 + d\mu^2 + e\mu\lambda + f\lambda + g\mu = 0.$$

By writing

$$bx - a\varkappa z \equiv l_1, \quad \varkappa^2 y(cy + fz) \equiv yu, \quad \varkappa z^2 - x^2 - \varkappa y \equiv f_2,$$
$$a\varkappa^2 zy - 2\varkappa bxy + cx^2 + d\varkappa^2 z^2 - e\varkappa xz \equiv \phi_2,$$
$$b\varkappa^2 y^2 - 2\varkappa cxy + e\varkappa^2 yz - f\varkappa xz + g\varkappa^2 z^2 \equiv \psi_2,$$

the equation of the scroll may be expressed by the vanishing of a determinant,

$$\begin{vmatrix} l_1 & \phi_2 & \psi_2 & yu & 0 \\ 0 & xl_1 & \phi_2 & \psi_2 & u \\ w & f_2 & \varkappa xy & 0 & 0 \\ 0 & xw & f_2 & \varkappa xy & 0 \\ 0 & 0 & xw & f_2 & \varkappa x \end{vmatrix} = 0,$$

which still contains the extraneous factor z^2. The residual curve is of order 5 and is cut by every generator twice. A multiple generator cannot appear. Type XVII.

The double quintic may break up into a quartic and a double directrix line. For, express the condition that any generator should cut a given line. The resultant is a (2, 2) correspondence between λ, μ of the kind here treated, but with restricted coefficients. The quartic curve cuts the double directrix in two points and every generator cuts it in one point. The scroll may be defined as generated by all the common secants of the line and the two conics. The quartic curve cannot further degrade, for, if a double generator appeared, the scroll would be unicursal; if a second directrix line were present, the scroll would belong to a linear congruence and consequently have no nodal curve. Type XVIII.

The scroll may have a third double conic, which may be determined as follows: Suppose $Ax + By + Cz = 0$ is the plane of the new conic, which latter will also pass through the origin. Solve for the point in which a variable generator cuts this plane,

$$x = \frac{-\lambda\mu\,(\varkappa C + \varkappa B\lambda)}{\mu\,(\mu B + \varkappa A) - \lambda\,(\varkappa C + \varkappa B\lambda)}\,,\quad y = \dots\,,\quad z = \dots$$

Now, consider any quadric surface, with undetermined coefficients, which passes through the origin; impose the condition that the point just found also lies on this surface.

There are condition equations to solve for the unknowns linearly, and putting the values found for the coefficients in the relation between λ, μ, it becomes a (2, 2) correspondence of the kind here needed and not having a double element, hence the new conic is double.

The residual nodal curve is a cubic which is cut once by every generator. It cannot further degrade. If a double directrix line exists, the possibility of a third nodal conic is excluded. The surface may be generated by the common secants of three conics, all of which pass through one point and each pair having one further point common. Type XIX.

10. A sextic scroll exists having a double conic and a residual curve of order 7, which is cut in three points by each generator. Take a binodal quartic in $z = 0$.

$$f(x, y, w) = (ax^2 + bxw + cw^2)\, y^2 + (a'x^2 + b'xw + c'w^2)\, yw + (a''x^2 + b''xw + c''w^2)\, w^2 = 0,$$

and a pencil of conics $\phi_2 + \varkappa\psi_2 = 0$ passing through the two nodes and two other fixed points of the quartic.

Make the conics of the pencil projective with the points of the conic $x = 0$, $yz = w^2$ such that the nodes, considered as points on the new conic correspond to conics of the pencil which touch one of the branches of the quartic at the respective node. The lines joining corresponding points will generate a scroll of the type desired. Type XX.

§3.—*General* (2, 4) *Correspondence.*

11. In No. 10, if the conic $x = 0$, $yz = w^2$ be replaced by the range

$$x = 0, \quad y = 0, \quad z = \varkappa w,$$

the line will be a double directrix. The residual curve is of order 8 which cuts the double directrix twice and each generator three times. The equation can be written directly by eliminating x_1, y, $\varkappa$, w_1 between the equations

$$\frac{x}{x_1} = \frac{y}{y_1} = \frac{\varkappa w - z}{\varkappa}, \quad f(x_1, y_1, w_1) = 0, \quad \phi_2(x_1, y_1, w_1) + \varkappa\psi_2(x_1, y_1, w_1) = 0,$$

and dividing out six extraneous linear factors $x^2 = 0$, $y^2 = 0$ and the first members of the equations of the planes containing the line x, y and one or the other of the basis points of the pencil of conics. Type XXI.

If $c'' = 0$, the quartic intersects the multiple directrix which now becomes a triple line composed of a double directrix and single generator. The residual curve is a sextic which cuts the triple line three times and every other generator twice. Type XXII.

Suppose now that the line be

$$x = 0, \quad w = 0, \quad z = \varkappa y.$$

Then it becomes a double directrix and double generator. The residual curve is a twisted cubic which cuts the multiple directrix twice but every other generator once. The scroll may be generated by lines joining points of a (2, 2) correspondence between a twisted cubic and one of its double secants, the points of intersection both being simple self-corresponding elements. Type XXIII.

Now, let one of the basis points of the pencil of conics be taken off the quartic. The line x, y will become a triple directrix. Let the conic which is tangent to one of the branches at the node correspond to the same point on the directrix. The latter is triple directrix and simple generator. The residual nodal curve is a twisted cubic which cuts the multiple line twice and every other generator once. Type XXIV. This form can be generated by means of a (2, 3) correspondence between a twisted cubic and a double secant, one point of intersection being doubly self-corresponding, the other simply.

Now, suppose the quartic to have a cusp at x, w. Let the pencil of conics have two basis points not on the quartic, and let the conic which touches the cuspidal tangent at the cusp correspond to the same point on the directrix. The latter becomes a fourfold directrix and the residual curve is a twisted cubic cutting the directrix and each generator once. The surface may be generated by means of a (2, 4) correspondence between a twisted cubic and one of its double secants, the points of intersection both being self-corresponding double points. Type XXV.

An illustration is furnished by the curve

$$x = \lambda(\lambda - 1), \quad y = \lambda^2(\lambda - 1), \quad z = \lambda$$

and the line

$$x = 0, \qquad y = 0, \qquad z = \mu.$$

From the equations of the line joining the point μ to the point λ one obtains

$$\lambda = \frac{y}{x}, \quad \mu = \frac{y\,(x^2 + z(y - x))}{x^3 + yw\,(y - x)}.$$

The (2, 2) correspondence between λ, μ must be satisfied by (0, 0) and by (1, 1); similar restrictions exist for the other forms.

12. If, in forms XXI to XXV, the correspondence were established by means of a pencil of lines instead of conics, a double generator would exist. Let the quartic be defined as in No. 10 and the points on the lines x, y be projectively associated with the lines

$$z = 0, \quad x + \mu w = 0,$$

wherein

$$\varkappa = \frac{\alpha\mu + \beta}{\gamma\mu + \delta}.$$

The elimination of $\varkappa$ gives

$$ay^2u^4 + (by^2 + a'xy)\,\mu^3 + (cy^2 + b'xy + a''x^2)\,\mu + (c''xy + b''x^2)\,\mu + c''x^2 = 0,$$
$$(\gamma z - \alpha w)\,\mu^2 + (\delta z + \alpha x - \beta w)\,\mu + \beta x = 0.$$

The factor x^2 can be removed from the μ eliminant, the other factor of which defines the surface.

The line $y = 0$, $\gamma z - \alpha w = 0$, corresponding to the line joining the two nodes, is a double generator. The residual nodal curve is of order 7, it cuts the double directrix once and each generator three times. Type XXVI.

When $c'' = 0$, the quartic intersects the multiple directrix, which now becomes a simple generator. The residual curve is a quintic which cuts the multiple directrix twice and each generator twice. Type XXVII.

If, in the general case $(c'' \neq 0)$ $\beta = 0$, the line

$$\gamma z - \alpha w = 0, \quad \delta z + \alpha x = 0$$

is a fourfold directrix; the residual nodal curve is composed of the two double generators $x = 0$, $z = 0$; $y = 0$, $\gamma z - \alpha w = 0$. This is the most general elliptic (2, 4) scroll which is contained in a linear congruence. The equation may be more easily written in the form

$$f(\lambda^4, \mu^2) = 0, \quad \lambda = \frac{y}{x}, \quad \mu = \frac{z}{w},$$

in which f has two finite double points.* Type XXVIII.

If the two directrices coincide, the equation may be written, if now $\lambda = \dfrac{y}{x}$, $\mu = \dfrac{xz - ayw}{xw}$,

$$\phi\,(\lambda, \mu) = \sum_{r=0}^{3} f_{6-2r}(\lambda)\,\mu^r = 0,$$

and ϕ has two finite double points. Type XXIX.

§4.—*Scrolls having a Plane Double Cubic.*

13. Consider a nodal cubic and a straight line passing through the node but not lying in the plane of the cubic. Let their points, which can be rationally

* See Journal, Vol. XXIII, p. 166, and Bulletin American Math. Society, Vol. 5, p. 343.

expressed in terms of parameters, be put in (2, 2) correspondence in such a way that both values of λ defining the node of the cubic correspond to the same point regarded as a point on the line.

Let $z = 0$, $axyw + f_3(x, y) = 0$ be the equations of the cubic, and $x = 0$, $y = 0$ be those of the line. Then

$$\frac{y}{x} = \lambda, \quad \frac{axyz}{axyw + f_3(x, y)} = \mu$$

are to be put in (2, 2) correspondence of the form

$$f_2(\lambda)\mu^2 + \phi_2(\lambda)\mu + \varkappa\lambda = 0,$$

which gives for the equation of the surface

$$f_2(x, y)\, a^2xyz^2 + \phi_2(x, y)\, az\left(axyw + f_3(x, y)\right) + \varkappa\left(axyw + f_3(x, y)\right)^2 = 0,$$

The line x, y is a double directrix and a double generator, two other generators passing through each point. The fourfold line and the nodal cubic constitute the whole of the double curve. Type XXX.

Now, consider a (3, 2) correspondence between the same elements which is restricted to the form

$$\phi_3(\lambda)\mu^2 + \lambda\mu\phi_2(\lambda) + \varkappa\lambda^2 = 0.$$

The equation of the surface is

$$\phi_3(x, y)\, a^2xz^2 + az\phi_2(x, y)\left(axyw + f_3(x, y)\right) + \varkappa\left(axyw + f_3(x, y)\right)^2 = 0.$$

This only differs from the preceding case in the form of ϕ_3; the previous type is derived from this one when the coefficient of x^3 is zero. The configuration of nodal lines is the same in the two cases, but the latter is an essentially different type, because three generators distinct from the line itself issue from each point of the multiple directrix. The fourfold line is now triple directrix and simple generator. Type XXXI.

If, finally, λ, μ be connected by a (4, 2) correspondence of the restricted form

$$\phi_4(\lambda)\mu^2 + \lambda\mu\phi_2(\lambda) + \varkappa\lambda^2 = 0,$$

the equation of the surface becomes

$$\phi_4(x, y)\, a^2z^2 + az\phi_2(x, y)\left(axyw + f_3(x, y)\right) + \varkappa\left(axyw + f_3(x, y)\right)^2 = 0.$$

In this case four generators issue from each point of the multiple directrix which is itself not a generator. Type XXXII.

If $\mu = \dfrac{ax^2z}{ax^2w + f_3(x, y)}$, the plane double cubic has a cusp at the point in which it meets the directrix line.

§5.—*Table of Forms of Double Curves.*

I. c_9^2.

II. $c_8^2 + g^2$.

III. $2d^3 + g^3$.

IV. $d^3 \equiv d'^3 + g^3$.

V. $c_6^2 + d^3$.

VI. $c_5^2 + d^3 + g^2$.

VII. $c_4^2 + d^3 + 2g^2$.

VIII. $2d^3 + 3g^2$.

IX. $d^3 \equiv d'^3 + 3g^2$.

X. $c_2^3 + (d^2, g)$.

XI. $c_2^3 + d^3$.

XII. $c_2^2 + g^2 + (d^2, g^2)$ (tacnodal).

XIII. $c_2^2 + g^2 + (d^3, g)$.

XIV. $c_2^2 + g^2 + d^4$.

XV. $c_2^2 + c_6^2 + d^2$.

XVI. $c_2^2 + c_5^2 + g^2 + d^2$.

XVII. $2c_2^2 + c_5^2$.

XVIII. $d^2 + 2c_2^2 + c_4^2$.

XIX. $3c_2^2 + c_3^2$ (skew).

XX. $c_2^2 + c_7^2$.

XXI. $d^2 + c_8^2$.

XXII. $c_6^2 + (d^2, g)$.

XXIII. c_3^2 (skew) $+ (d^2, g^2)$.

XXIV. c_3^2 (skew) $+ (d^3, g)$.

XXV. c_3^2 (skew) $+ d^4$.

XXVI. $c_7^2 + d^2 + g^2$.

XXVII. $c_5^2 + g^2 + (d^2, g)$.

XXVIII. $d^4 + d^2 + 2g^2$.

XXIX. $d^4 \equiv (d^2, g^2) + 2g^2$.

XXX. $(d^2, g^2) + c_3^2$ (plane).

XXXI. $(d^3, g) + c_3^2$ (plane).

XXXII. $d^4 + c_3^2$ (plane).

CORNELL UNIVERSITY, *June*, 1902.